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The Equilibrium States for a Model with Two Kinds of Bose Condensation

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We study the equilibrium Gibbs states for a Boson gas model, defined by Bru and Zagrebnov, which has two phase transitions of the Bose condensation type. The two phase transitions correspond to two distinct mechanisms by which these condensations can occur. The first (*non-conventional*) Bose condensation is mediated by a zero-mode interaction term in the Hamiltonian. The second is a transition due to saturation quite similar to the *conventional* Bose–Einstein (BE) condensation in the ideal Bose gas. Due to repulsive interaction in non-zero modes the model manifests a generalized type III; *i.e., non-extensive* BE condensation. Our main result is that, as in the ideal Bose gas, the conventional condensation is accompanied by a loss of strong equivalence of the canonical and grand canonical ensembles whereas the non-conventional one, due to the interaction, does not break the equivalence of ensembles, at least not on the level of the gauge invariant states. It is also interesting to note that the type of (generalized) condensate, I, II, or III (in the terminology of van den Berg, Lewis, and Pulé), has no effect on the equivalence of ensembles. These results are proved by computing the generating functional of the cyclic representation of the Canonical Commutation Relation (CCR) for the corresponding equilibrium Gibbs states.

KEY WORDS: Quantum equilibrium states; generating functional; Bose condensation; Canonical Commutation Relations (CCR); equivalence of ensembles.

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1. INTRODUCTION AND SETUP OF THE PROBLEM

In recent years, the phenomenon of Bose condensation, described first by Einstein in 1925,[1] has become an active area of research, both experimentally and theoretically. An example is the existence of a new kind of condensation which was recently theoretically discovered by an analysis of the thermodynamic behaviour of the Bogoliubov Weakly Imperfect Bose Gas[2–7] or of some specific Bose systems with diagonal interactions.[8, 9] This new Bose condensation, denoted as *non-conventional* Bose condensation, is in fact induced by a mechanism of *interaction* whereas the *conventional* one; i.e., the Bose–Einstein (BE) condensation, appears by a phenomenon of *saturation*; i.e., by the existence only of a *bounded* critical density.[10–18] In fact, the Bose condensation occurring in the Huang–Yang–Luttinger model and in the, so-called, Full Diagonal Model, studied in great detail in refs. 19–21, should also be considered as examples of the *non-conventional* type, since in the both cases it is due to the interaction in those models.

The analysis of the effect of the *conventional* BE condensation on the equilibrium states was initially worked out by Araki and Woods in the case of the Perfect Bose Gas (PBG),[22] and further refined in refs. 23–25. A well-known model that exhibits *non-conventional* condensation is the Bogoliubov model.[2–5, 7] As a complete and rigorous analysis of the Gibbs states of the Bogoliubov model is beyond the reach of current techniques, we propose to analyze the effect of the *non-conventional* Bose condensation on the Gibbs states in the simpler model defined in ref. 8, see (1.1), in which the both kinds of Bose condensation occur.

The model we consider is a system of spinless bosons of mass m enclosed in a cubic box $\Lambda \subset \mathbb{R}^d$ of volume $V = |\Lambda| = L^d$ centered at the origin with a Hamiltonian of the form

$$H^I_\Lambda = T_\Lambda + U^0_\Lambda + U^I_\Lambda = H^0_\Lambda + U^I_\Lambda, \tag{1.1}$$

with

$$\begin{aligned}
T_\Lambda &= \sum_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k a^*_k a_k,\ \varepsilon_k = \hbar^2 k^2/2m, \qquad \text{for all} \quad k \neq 0,\\
U^0_\Lambda &= \varepsilon_0 a^*_0 a_0 + \frac{g_0}{2V} a^*_0 a^*_0 a_0 a_0, \qquad \varepsilon_0 < 0, \quad g_0 > 0, \qquad (1.2)\\
U^I_\Lambda &= \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} g_k a^*_k a^*_k a_k a_k, \qquad g_+ \geqslant g_k \geqslant g_- > 0.
\end{aligned}$$

The sums run over the set

$$\Lambda^* = \left\{ k \in \mathbb{R}^d : k_\alpha = \frac{2\pi n_\alpha}{L}, \, n_\alpha = 0, \, \pm 1, \, \pm 2, ..., \, \alpha = 1, 2, ..., \, d \right\},$$

i.e., we consider *periodic boundary conditions* on $\partial \Lambda$. We denote the corresponding one-particle Hilbert space by $L^2(\Lambda)$. Here $a_k^\# = \{a_k \text{ or } a_k^*\}$ are the usual boson creation and annihilation operators for the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} e^{ikx}$, $k \in \Lambda^*$, $x \in \Lambda$, acting on the boson Fock space $\mathscr{F}_\Lambda^B \equiv \mathscr{F}^B(L^2(\Lambda))$ over $L^2(\Lambda)$:

$$\mathscr{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathscr{H}_B^{(n)}, \tag{1.3}$$

where

$$\mathscr{H}_B^{(n)} \equiv (L^2(\Lambda^n))_{\text{symm}} \tag{1.4}$$

is the *symmetrized* n-particle Hilbert spaces appropriate for bosons, and $\mathscr{H}_B^{(0)} = \mathbb{C}$. We denote by

$$A^{(n)} \equiv A \upharpoonright \mathscr{H}_B^{(n)}$$

the restriction of an operator A acting on the boson Fock space $\mathscr{F}_\Lambda^B$ to $\mathscr{H}_B^{(n)}$.

This Bose model was introduced and studied in ref. 8. Clearly, it can be considered as a perturbation of the kinetic-energy T_Λ with diagonal interactions in modes $k = 0$ (U_Λ^0) and $k \neq 0$ (U_Λ^I).

The main interest of this model is that it exhibits *two* phase transitions accompanied by the formation of *non-conventional* and *conventional* Bose condensation. The first is due to the *negative* effective excitation energy $\varepsilon_0 < 0$, which leads to a macroscopic occupation of the zero-mode in some interval of negative chemical potentials. This condensation occurs at any temperature. The second is a conventional condensation due to *saturation*. Notice that the second repulsive term ($g_0 > 0$) in U_Λ^0 prevents the Bose gas from collapse; i.e., it keeps the particle density finite for negative chemical potentials.

We now summarize the main results of ref. 8, where it was shown in detail that the model H_Λ^I (1.1) displays a two-stage Bose condensation. Let μ and $\theta = \beta^{-1}$ denote the chemical potential and temperature, respectively. Furthermore

$$N_\Lambda = \sum_{k \in \Lambda^*} N_k \equiv \sum_{k \in \Lambda^*} a_k^* a_k$$

is the particle-number operator and $\langle - \rangle_{H_\Lambda^I}(\beta, \mu)$ represents the grand-canonical Gibbs state for the Hamiltonian H_Λ^I. Define

$$\rho^P(\beta, \mu) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (e^{\beta(\varepsilon_k - \mu)} - 1)^{-1}\, dk \tag{1.5}$$

and

$$\rho_c^P(\beta) \equiv \sup_{\mu < 0} \rho^P(\beta, \mu) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (e^{\beta\varepsilon_k} - 1)^{-1}\, dk < +\infty, \qquad d > 2. \tag{1.6}$$

ρ^P and ρ_c^P are the density and the critical density of the PBG, respectively. The following results are proved in ref. 8:

- The model has well-behaved thermodynamics; i.e., the pressure exists, for the temperature $\theta = \beta^{-1} \geqslant 0$, and chemical potential $\mu \leqslant 0$. We denote this domain by $Q = \{(\theta, \mu) : \theta \geqslant 0, \mu \leqslant 0\}$. Notice that the same is valid for any finite $\varepsilon_0 \in \mathbb{R}^1$.

- There is no condensation for $\mu \leqslant \varepsilon_0$, but condensation occurs for $\varepsilon_0 < \mu \leqslant 0$. More precisely, one has a macroscopic occupation of the $k = 0$ mode, given by

$$\rho_0^I(\beta, \mu) \equiv \lim_\Lambda \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu) = \max\left\{0, \frac{\mu - \varepsilon_0}{g_0}\right\}, \tag{1.7}$$

i.e., there is Bose condensation due to the instability implied by the negative excitation energy $\varepsilon_0 < 0$, thought of as being induced by an interaction mechanism (*non-conventional* condensation). Note that the density of the condensate depends linearly on $\mu - \varepsilon$ and not on β.

- For $d > 2$, the non-conventional Bose condensate density $\rho_0^I(\beta, \mu)$ (1.7) and the total particle density

$$\rho^I(\beta, \mu) \equiv \lim_\Lambda \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu) = \rho^P(\beta, \mu) + \rho_0^I(\beta, \mu) \tag{1.8}$$

attain their maxima at $\mu = 0$. For densities exceeding a critical value,

$$\rho > \rho_c^I(\beta) \equiv \sup_{\mu \leqslant 0} \rho^I(\beta, \mu) = \lim_{\mu \to 0^-} \rho^I(\beta, \mu) = \rho_c^P(\beta) - \frac{\varepsilon_0}{g_0} < +\infty, \tag{1.9}$$

the H_Λ^I model (1.1) manifests a generalized type III (i.e., *non-extensive*) conventional BE condensation:

$$\tilde{\rho}_0^I(\beta, \rho) \equiv \lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^* : 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_\Lambda^I}$$
$$= \begin{cases} 0 & \text{for} \quad \rho \leqslant \rho_c^I(\beta) \\ \rho - \rho_c^I(\beta) & \text{for} \quad \rho > \rho_c^I(\beta). \end{cases} \tag{1.10}$$

The term *non-extensive* refers to the fact that no single mode has a macroscopic occupation of particles. For $\varepsilon_0 < 0$ this conventional condensate *coexists* at $\mu = 0$ with the non-conventional condensate $\rho_0^I(\beta, \mu = 0)$ in the mode $k = 0$. Notice that the conventional BE condensation (1.10) appears in spite of the *repulsive* interaction U_Λ^I (1.2) between bosons in modes $k \neq 0$. But it is because of this repulsion that the condensation is non-extensive.

- Remark that formula (1.7) is also valid for $\varepsilon_0 \geqslant 0$. In this case one gets $\rho_0^I(\beta, \mu) \equiv 0$; i.e., the model (1.1) manifests, for $d > 2$, only the non-extensive conventional condensation (1.10).

More details about the non-extensive BE condensation one finds in Appendix A of the present paper. These results are an extension of those of ref. 8. They are indispensable for calculating the generating functional for the model (1.1).

We conclude this introduction with a few remarks. The first concerns the effect of the repulsive term U_Λ^I in (1.2). It is known that for $g_- > 0$, this interaction converts the *conventional* condensation from *type I* (macroscopic occupation of bounded number of modes $k \neq 0$), e.g., a single mode $k = 0$ such as occurs in the PBG, $\varepsilon_0 = g_0 = g_+ = 0$), into one of *type III* (no macroscopic occupation of a single mode, accumulation of a finite fraction of the particles in an infinitesimal interval near $k = 0$).[6, 8, 18] The simplest example corresponds to the PBG ($\varepsilon_0 = g_0 = g_+ = 0$)in an *isotropically* dilated container, when the macroscopic occupation of the single mode $k = 0$ is transformed by the pure repulsive interaction, $g_0 > 0$, $g_- > 0$, $\varepsilon_0 = 0$, into a non-extensive BE condensation.[6] We stress here the *isotropic* shape of the container, since the *conventional* condensation is so subtle matter that the PBG itself manifests non-extensive BE condensation, if for example, considered in a dilated rectangular box with highly anisotropic growth rates for the edges.[6, 12, 14, 15]

Our second remark concerns the *dimension* dependence of the phase transitions. In contrast to the *conventional* condensation caused by saturation for $d > 2$, the *non-conventional* condensation (1.7) is due to interaction, and it exists for all dimensions, including $d = 1$, and 2. This is another indication that the simplified model (1.1) with diagonal interaction is similar to the Bogoliubov Gas.[5–7] Moreover, in contrast to conventional BE condensations, the non-conventional condensation may emerge as a first-order phase transitions: the Bose condensate density appears discontinuously, see for example the thermodynamic behaviour of the Bogoliubov gas,[2–4, 7] or the Huang–Yang–Luttinger model, see refs. 19 and 20 and ref. 26. This transition in the simplified model studied here is continuous however, see (1.7).

The bulk of the present paper is devoted to the study of the Gibbs states of the model (1.1). In particular we shall calculate the generating functional of the cyclic representation of the Canonical Commutation Relations (CCR) for the Gibbs states of the model (1.1), a method introduced in 1963 by Araki and Woods.[22] From the generating functional it is then straightforward to read off properties such as the breakdown of the strong equivalence of ensembles as was done in refs. 23–25 and 27. We shall see that the two phase transitions have their distinct effects on the generating functional.

Before we embark on the actual calculation we present, in Section 2, the relevant known properties of the generating functional of the cyclic representation of the CCR for the Gibbs states of the PBG.[27] In Section 3 we calculate in the thermodynamic limit the grand-canonical generating functional for the Gibbs state $\langle - \rangle_{H_\Lambda^I}(\beta, \mu)$ associated with the model (1.1) for a fixed chemical potential $\mu < 0$, or a fixed density $\rho < \rho_c^I(\beta)$. In the next Section 4 we determine the generating functional for a fixed particle density $\rho \geqslant \rho_c^I(\beta)$ (1.9). In Section 5 we summarize our conclusions and formulate some tentative generalizations. Some technical results are collected in the Appendix A.

2. GENERATING FUNCTIONALS

The purpose of this section is to review the characterization of (Gibbs) states of a Bose system by their generating functional, a method originally introduced by Araki and Woods in the case of the PBG.[22] For each Gibbs state there is a representation of the Canonical Commutation Relations (CCR) given by the GNS construction. For a complete description see ref. 27, and also refs. 23–25 for a detailed analysis of the PBG Gibbs state. Here, we only present a quick overview.

Let M be a complex pre-Hilbert space with the corresponding scalar product $(.,.)_M$. We consider a representation of the CCR over M given by a map $h \mapsto W(h)$ from M to a space $U(\mathscr{H})$ of unitary operators on a Hilbert space $\mathscr{H}$ satisfying

$$W(h_1)\, W(h_2) = \exp\left\{ -\frac{i}{2} \operatorname{Im}(h_1, h_2)_M \right\} W(h_1 + h_2), \tag{2.1}$$

and such that the map $\lambda \mapsto W(\lambda h)$ from $\mathbb{R}$ to $U(\mathscr{H})$ is strongly continuous. By Stone's theorem,[27] the continuity implies the existence of self-adjoint operators $R(h)$ such that

$$W(h) = \exp\{iR(h)\}. \tag{2.2}$$

The $R(h)$ are called the *field operators* and can be interpreted as the random variables of a non-commutative probability theory, since by (2.1) one gets

$$[R(h_1), R(h_2)] = i \operatorname{Im}(h_1, h_2)_M. \tag{2.3}$$

Note that the map $h \mapsto R(h)$ is a linear over $\mathbb{R}$. For $h \in M$, we can now define the *creation* and *annihilation* operators $a^*(h)$ and $a(h) \equiv (a^*(h))^*$ by

$$a^*(h) \equiv \frac{1}{\sqrt{2}} \{R(h) - iR(ih)\}, \qquad a(h) \equiv \frac{1}{\sqrt{2}} \{R(h) + iR(ih)\}. \tag{2.4}$$

A representation of the CCR is called *cyclic* if there is a vector Ω in $\mathscr{H}$ such that the set $\{W(h)\, \Omega\}_{h \in M}$ is dense in $\mathscr{H}$. Such Ω is called a cyclic vector. It can be shown that, for every *regular* Gibbs state $\langle \cdot \rangle$, there is unique (up to unitary equivalence) representation of the CCR with cyclic vector Ω such that

$$\langle \exp\{iR(h)\} \rangle = (\Omega, W(h)\, \Omega)_{\mathscr{H}}.$$

The *generating functional* of the representation is defined by

$$\mathbb{E}(h) \equiv (\Omega, W(h)\, \Omega)_{\mathscr{H}}, \qquad h \in M. \tag{2.5}$$

The generating functional plays the same rôle for a state on the CCR algebra, as the characteristic function for probability distribution, see ref. 27.

Theorem 2.1 (Araki–Segal). Let $\mathbb{E}$ be the generating functional of a cyclic representation of the CCR over M. Then it satisfies:

(i) $\mathbb{E}(0) = 1$;

(ii) for any finite set $\{c_j \in \mathbb{C}; h_j \in M\}$, one has

$$\sum_{l,s=1}^{n} \mathbb{E}(h_l - h_s) \exp\left\{\frac{i}{2} \operatorname{Im}(h_l, h_s)_M\right\} \overline{c_l} c_s \geqslant 0;$$

(iii) for $h \in M$, the map $\lambda \to \mathbb{E}(\lambda h)$ from $\mathbb{C}$ to $\mathbb{R}$ is continuous.

Conversely, any generating functional $\mathbb{E}: M \to \mathbb{C}$ satisfying (i), (ii), and (iii) is a generating functional of a cyclic representation of the CCR.

Our concrete setup will be as follows. For a (sufficiently regular) *finite* volume, $\Lambda \subset \mathbb{R}^d$, the grand canonical Gibbs state $\langle \cdot \rangle_\Lambda (\beta, \mu)$, is defined on the set of bounded operators acting on the boson Fock space $\mathscr{F}_\Lambda^B \equiv \mathscr{F}^B(L^2(\Lambda))$ over $L^2(\Lambda)$, see (1.3). In order to analyze the state $\langle \cdot \rangle_\Lambda (\beta, \mu)$, we use the Fock representation $W^{\mathscr{F}_\Lambda^B}$ of the CCR over the pre-Hilbert space $M = \mathscr{D}_\Lambda$ (the space of the $C_0^\infty(\Lambda)$-functions with compact supports contained in Λ). Its generating functional (2.5) is equal to $\mathbb{E}_{\mathscr{F}_\Lambda^B}(h) = e^{-\frac{1}{4}\|h\|^2}$, where cyclic vector Ω is vacuum in $\mathscr{H} = \mathscr{F}_\Lambda^B$: $a(h)\,\Omega = 0$ for any $h \in \mathscr{D}_\Lambda$. Since $\mathscr{D}_\Lambda$ is dense in $L^2(\Lambda)$, one can extend $W^{\mathscr{F}_\Lambda^B}$ to the later. We shall calculate the generating functional

$$\mathbb{E}_\Lambda(\beta, \mu; h) \equiv \langle W^{\mathscr{F}_\Lambda^B}(h) \rangle_\Lambda (\beta, \mu), \qquad h \in \mathscr{D}_\Lambda, \tag{2.6}$$

and study its thermodynamic limit ($\Lambda \uparrow \mathbb{R}^d$).

3. GIBBS STATE AND NON-CONVENTIONAL CONDENSATION

Recall from Section 1 that condensation in the exactly solvable model H_Λ^I (1.1) occurs in two stages: for intermediate densities $\rho < \rho_c^I(\beta)$; i.e., for negative chemical potentials $\varepsilon_0 < \mu < 0$, one has only *non-conventional* Bose condensation in the $k = 0$ mode due to the diagonal perturbation U_Λ^0 (1.2) of the PBG (cf. (1.7)), whereas for large densities $\rho \geqslant \rho_c^I(\beta)$ $(\mu = 0)$, this condensate coexists with conventional (type III) generalized BE condensation corresponding to the standard mechanism of saturation, see (1.9) and (1.10).

In this section we study the influence on the corresponding Gibbs state of the first stage of condensation: the *non-conventional* one (1.7) which appears for a fixed chemical potential $\varepsilon_0 < \mu \leqslant 0$. Following refs. 22–25, we use the Fock representations of the CCR[27] over the space $\mathscr{D}_\Lambda$ of C^∞-smooth functions with compact support contained in Λ (Section 2) and we define by

$$\mathbb{E}_\Lambda^I(\beta, \mu; h) \equiv \langle W^{\mathscr{F}_\Lambda^B}(h) \rangle_{H_\Lambda^I} (\beta, \mu), \qquad h \in \mathscr{D}_\Lambda, \tag{3.1}$$

the grand-canonical generating functional of the model (1.1):

$$H_\Lambda^I = \sum_{k\in\Lambda^*} \left\{ \varepsilon_k a_k^* a_k + \frac{g_k}{2V} a_k^{*^2} a_k^2 \right\} = \sum_{k\in\Lambda^*} H_k^I,$$
$$\varepsilon_{k\neq 0} = \hbar^2 k^2/2m \geqslant 0, \quad \varepsilon_0 < 0. \tag{3.2}$$

Here the operators $W^{\mathscr{F}_\Lambda^B}(h)$ for $h \in \mathscr{D}_\Lambda$, are defined by (2.2)–(2.4).

Note that the boson Fock space $\mathscr{F}_\Lambda^B$ (1.3) is isomorphic to the tensor product: $\mathscr{F}_\Lambda^B \approx \bigotimes_{k\in\Lambda^*} \mathscr{F}_k^B$, where $\mathscr{F}_k^B \equiv \mathscr{F}^B(\mathscr{H}_k)$ is the boson Fock space constructed on the one-dimensional Hilbert space

$$\mathscr{H}_k = \{\lambda e^{ikx}\}_{\lambda\in\mathbb{C}}. \tag{3.3}$$

Then using the Fourier decomposition

$$a^*(h) = \int_\Lambda dx h(x)\, a^*(x) = \frac{1}{\sqrt{V}} \sum_{k\in\Lambda^*} (e^{ikx}, h)_{L^2(\Lambda)}\, a^*(\psi_k) \equiv \frac{1}{\sqrt{V}} \sum_{k\in\Lambda^*} h_k a_k^*,$$
$$a(h) = \int_\Lambda dx \overline{h(x)}\, a(x) = \frac{1}{\sqrt{V}} \sum_{k\in\Lambda^*} (h, e^{ikx})_{L^2(\Lambda)}\, a(\psi_k) \equiv \frac{1}{\sqrt{V}} \sum_{k\in\Lambda^*} \overline{h_k} a_k, \tag{3.4}$$

we can write the generating functional $\mathbb{E}_\Lambda^I(\beta, \mu; h)$ (3.1) in the following form:

$$\begin{aligned}\mathbb{E}_\Lambda^I(\beta, \mu; h) &= \prod_{k\in\Lambda^*} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_\Lambda^I}(\beta, \mu) \\ &= \prod_{k\in\Lambda^*} \frac{\mathrm{Tr}_{\mathscr{F}_k^B}(e^{-\beta H_k^I(\mu)} e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)})}{\mathrm{Tr}_{\mathscr{F}_k^B}(e^{-\beta H_k^I(\mu)})} \\ &= \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{H_0^I} \prod_{k\in\Lambda^*\setminus\{0\}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_k^I}.\end{aligned} \tag{3.5}$$

Next, we study the two factors corresponding to cases $k = 0$ and $k \neq 0$ separately. Denote by $\mathscr{D} = \bigcup_{\Lambda\subset\mathbb{R}^d} \mathscr{D}_\Lambda$ the space of C^∞-smooth functions on $\mathbb{R}^d$ having compact support, and by $\hat{h}_k \equiv (e^{ikx}, h)_{L^2(\mathbb{R}^d)}$, $k \in \mathbb{R}^d$.

Theorem 3.1. Let $\varepsilon_0 \in \mathbb{R}^1$ and $g_0 > 0$. Suppose that $0 \leqslant g_- \leqslant g_k \leqslant g_+$ for $k \in \Lambda^*\setminus\{0\}$. Then for $\mu < 0$ and any h in the space $\mathscr{D}$ one gets that:

(i) for the mode $k = 0$

$$\lim_\Lambda \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{H_0^I} = J_0(\sqrt{2\rho_0^I(\beta, \mu)}\, |\hat{h}_0|), \tag{3.6}$$

where the non-conventional Bose-condensate density $\rho_0^I(\beta, \mu)$ is defined by (1.7);

(ii) for the second factor in (3.5) we have

$$\lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_k^I} = \exp\{-\tfrac{1}{4}\|h\|^2 - \tfrac{1}{2} A_{\beta,\mu}(h, h)\}, \tag{3.7}$$

where the sesquilinear form $A_{\beta,\mu}(u, v)$, for $u, v \in \mathscr{D}$, is defined by

$$A_{\beta,\mu}(u, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\overline{\hat{u}_k} \hat{v}_k}{e^{\beta(\varepsilon_k - \mu)} - 1} \, dk. \tag{3.8}$$

Proof. (i) Let $\{\psi_n\}_{n \geqslant 0} \subset \mathscr{F}_0^B$ be an orthonormal base of eigenvectors of the operator $a_0^* a_0$:

$$a_0^* a_0 \psi_n = n \psi_n.$$

Then one gets

$$\begin{aligned} X &\equiv \mathrm{Tr}_{\mathscr{F}_0^B}(e^{-\beta(H_0^I - \mu N_0)} e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)}) \\ &= \sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_0 - \mu - \frac{g_0}{2V}) n + \frac{g_0}{2V} n^2]} (\psi_n, e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \psi_n)_{\mathscr{F}_\Lambda^B}. \end{aligned} \tag{3.9}$$

By the Baker–Campbell–Hausdorff formula:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]}, \qquad \text{if} \quad [A, [A, B]] = [B, [A, B]] = 0, \tag{3.10}$$

we obtain that

$$\exp\left\{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)\right\} = e^{-\frac{1}{4V}|h_0|^2} \exp\left\{\frac{i}{\sqrt{2V}} h_0 a_0^*\right\} \exp\left\{\frac{i}{\sqrt{2V}} \overline{h_0} a_0\right\}. \tag{3.11}$$

Therefore, since

$$\begin{aligned} a_0 \psi_n &= \sqrt{n}\, \psi_{n-1}, \\ a_0^* \psi_n &= \sqrt{n+1}\, \psi_{n+1}, \\ (\psi_n, \psi_{n'})_{\mathscr{F}_\Lambda^B} &= \delta_{n,n'}, \end{aligned}$$

by (3.11) the trace (3.9) equals:

$$X = e^{-\frac{1}{4V}|h_0|^2} \sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_0-\mu-\frac{g_0}{2V})\,n+\frac{g_0}{2V}n^2]} \|e^{\frac{i}{\sqrt{2V}}\overline{h_0}a_0}\psi_n\|^2$$

$$= e^{-\frac{1}{4V}|h_0|^2} \sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_0-\mu-\frac{g_0}{2V})\,n+\frac{g_0}{2V}n^2]} \sum_{l=0}^{n} \left(-\frac{|h_0|^2}{2V}\right)^l \frac{n!}{(l!)^2(n-l)!}. \tag{3.12}$$

Using the Laguerre polynomials

$$L_n(z) \equiv \sum_{l=0}^{n} \frac{n!}{(l!)^2(n-l)!}(-z)^l, \qquad n \geqslant 0,$$

(3.12) can be rewritten to give

$$X = e^{-\frac{1}{4V}|h_0|^2} \sum_{n=0}^{+\infty} L_n\left(\frac{|h_0|^2}{2V}\right) e^{-\beta[(\varepsilon_0-\mu-\frac{g_0}{2V})\,n+\frac{g_0}{2V}n^2]}.$$

Consequently, we obtain

$$\langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0}a_0+h_0a_0^*)}\rangle_{H_0^I} = e^{-\frac{1}{4V}|h_0|^2} \frac{\sum_{n=0}^{+\infty} L_n\left(\frac{|h_0|^2}{2n}\frac{n}{V}\right) e^{-\beta V\,[(\varepsilon_0-\mu-\frac{g_0}{2V})\frac{n}{V}+\frac{g_0}{2}(\frac{n}{V})^2]}}{\sum_{n=0}^{+\infty} e^{-\beta V\,[(\varepsilon_0-\mu-\frac{g_0}{2V})\frac{n}{V}+\frac{g_0}{2}(\frac{n}{V})^2]}}. \tag{3.13}$$

Notice that the probability distributions in (3.13):

$$F_V(x) \equiv \frac{\sum_{0\leqslant n/V\leqslant x} e^{-\beta V\,[(\varepsilon_0-\mu-\frac{g_0}{2V})\frac{n}{V}+\frac{g_0}{2}(\frac{n}{V})^2]}}{\sum_{n=0}^{+\infty} e^{-\beta V\,[(\varepsilon_0-\mu-\frac{g_0}{2V})\frac{n}{V}+\frac{g_0}{2}(\frac{n}{V})^2]}} \tag{3.14}$$

satisfy the Laplace *large deviation principle*:[28, 29]

$$\lim_{\Lambda} \int_{\mathbb{R}^1} f(x)\,dF_V(x) = f\left(\max\left\{0, \frac{\mu-\varepsilon_0}{g_0}\right\}\right), \tag{3.15}$$

for any bounded continuous function f on $\mathbb{R}^1$.

The Laguerre polynomials have the property

$$\lim_{n\to+\infty} L_n(z/n) = J_0(2\sqrt{z}) = \sum_{l=0}^{\infty} \frac{1}{(l!)^2}(-z)^l, \qquad \text{for} \quad z \in \mathbb{C}, \tag{3.16}$$

as entire analytic functions in $\mathbb{C}$. Here $J_0(x)$ is the Bessel function of order 0. Using this and (3.15), we find the thermodynamic limit of (3.13) to be

$$\lim_{\Lambda} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{H_0^I} = J_0(\sqrt{2\rho_0^I(\beta, \mu)\, |\hat{h}_0|^2}),$$

for any $h \in \mathscr{D}$, since $h_0 = \hat{h}_0$ for Λ sufficiently large. Thus, by definition (1.7), we deduce (3.6).

(ii) Similar to the proof of (3.13) we get for $k \in \Lambda^* \setminus \{0\}$ that

$$\Gamma_k \equiv \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_k^I} = e^{-\frac{1}{4V}|h_k|^2} \frac{\sum_{n=0}^{+\infty} L_n\left(\frac{|h_k|^2}{2V}\right) e^{-\beta V\, [(\varepsilon_k - \mu - \frac{g_k}{2V})\frac{n}{V} + \frac{g_k}{2}(\frac{n}{V})^2]}}{\sum_{n=0}^{+\infty} e^{-\beta V\, [(\varepsilon_k - \mu - \frac{g_k}{2V})\frac{n}{V} + \frac{g_k}{2}(\frac{n}{V})^2]}}. \tag{3.17}$$

Let

$$\gamma_k \equiv e^{-\frac{1}{4V}|h_k|^2} \frac{\sum_{n=0}^{+\infty} L_n\left(\frac{|h_k|^2}{2V}\right) e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]}}{\sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]}}. \tag{3.18}$$

Since

$$\sum_{n=0}^{+\infty} L_n(z)\, s^n = \frac{1}{1-s} \exp\left\{-z \frac{s}{1-s}\right\}, \tag{3.19}$$

we readily get that

$$\lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \gamma_k = \exp\{-\tfrac{1}{4}\|h\|^2 - \tfrac{1}{2} A_{\beta,\mu}(h, h)\}, \tag{3.20}$$

see (3.7).

To show that one gets the same limit for $\lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \Gamma_k$ we define

$$\Gamma_k(t_k) \equiv e^{-\frac{1}{4V}|h_k|^2} \frac{\sum_{n=0}^{+\infty} L_n\left(\frac{|h_k|^2}{2V}\right) e^{-\beta V\, [(\varepsilon_k - \mu - \frac{g_k}{2V})\frac{n}{V} + \frac{t_k}{2}(\frac{n}{V})^2]}}{\sum_{n=0}^{+\infty} e^{-\beta V\, [(\varepsilon_k - \mu - \frac{g_k}{2V})\frac{n}{V} + \frac{t_k}{2}(\frac{n}{V})^2]}}. \tag{3.21}$$

Therefore, $\Gamma_k(t_k = 0) = \gamma_k$ and $\Gamma_k(t_k = g_k) = \Gamma_k$. Since $\Gamma_k(t_k) \in \mathscr{C}^\infty(\mathbb{R}^1_+)$ for each $k \in \Lambda^* \setminus \{0\}$, then to prove that $\lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \Gamma_k$ coincides with (3.20) it is sufficient to estimate the asymptotic behaviour of derivative

$$\partial_{t_k} \Gamma_k(t_k = 0) = \frac{\Delta_k(V)}{\{\sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]}\}^2}\, e^{-\frac{1}{4V}|h_k|^2}, \tag{3.22}$$

for $V \to \infty$. Here

$$\Delta_k(V) = \sum_{n=0}^{+\infty} \left[\left\{ L_n\left(\frac{|h_k|^2}{2V}\right) - 1 \right\} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \left(-\frac{\beta n^2}{2V} \right) \right] \sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]}$$
$$+ \sum_{n=0}^{+\infty} \left[\left\{ L_n\left(\frac{|h_k|^2}{2V}\right) - 1 \right\} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \right] \sum_{n=0}^{+\infty} \left[e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \left(\frac{\beta n^2}{2V} \right) \right]. \tag{3.23}$$

Since (3.16) implies the convergence of derivatives, one gets the estimate

$$\left| \frac{1}{n} L_n'\left(\frac{z}{m}\right) \right| \leqslant C_{z_0}, \qquad z \in [0, z_0], \tag{3.24}$$

for any $z_0 > 0$ and $n \leqslant m$. Therefore, in this domain we have:

$$\left| L_n\left(\frac{z}{m}\right) - 1 \right| \leqslant C_{z_0} n \frac{z}{m}. \tag{3.25}$$

Let $n_0(V) \equiv [V^{1-\delta}]$ for some $\delta \in (0, 1)$. Here $[x]$ denotes the integer part of the real x. Then, since $g_k \leqslant g_+$, by virtue of (3.25) we can find $C_k > 0$ such that for any $\mu < 0$ one gets the estimates:

$$\left| \sum_{n=0}^{n_0(V)} \left[\left\{ L_n\left(\frac{|h_k|^2}{2V}\right) - 1 \right\} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \left(-\frac{\beta n^2}{2V} \right) \right] \sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \right|$$
$$\leqslant C_k \frac{\beta\, |h_k|^2}{4V^2} [\partial_y^3 f(y = \beta(\varepsilon_k - \mu))]\, f(\beta(\varepsilon_k - \mu)), \tag{3.26}$$

and

$$\left| \sum_{n=0}^{n_0(V)} \left[\left\{ L_n\left(\frac{|h_k|^2}{2V}\right) - 1 \right\} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \right] \sum_{n=0}^{+\infty} \left[e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\, n]} \left(\frac{\beta n^2}{2V} \right) \right] \right|$$
$$\leqslant C_k \frac{\beta\, |h_k|^2}{4V^2} [\partial_y f(y = \beta(\varepsilon_k - \mu))][\partial_y^2 f(y = \beta(\varepsilon_k - \mu))], \tag{3.27}$$

where $f(\beta(\varepsilon_k - \mu)) = (1 - e^{-\beta(\varepsilon_k - \mu)})^{-1}$.

On the other hand, for large n the Laguerre polynomials have the following asymptotics:

$$L_n(x) = \frac{e^{x/2}}{(\pi^2 n x)^{1/4}} \cos[2\sqrt{nx} - \pi/4] + O(n^{-3/4}), \tag{3.28}$$

for $x > 0$,

$$L_n(x) = e^{x/2} J_0(2\sqrt{(n+1/2)\,x}) + O(n^{-3/4}), \tag{3.29}$$

for $a \leqslant x \leqslant b$, $a > 0$, and

$$L_n(x) = 1 - nx + O((nx)^2), \tag{3.30}$$

for $nx \to 0$. Therefore, we get, for some $D_1(k), D_2(k) > 0$, the estimates:

$$\left| \sum_{n > n_0(V)}^{+\infty} \left[\left\{ L_n\left(\frac{|h_k|^2}{2V}\right) - 1 \right\} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\,n]} \left(\frac{\beta n^2}{2V}\right) \right] \sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\,n]} \right|$$
$$\leqslant D_1(k)\, \frac{\beta (n_0(V))^{7/4}}{V^{3/4}}\, e^{-\beta(\varepsilon_k - \mu)\, n_0(V)} f(\beta(\varepsilon_k - \mu)), \tag{3.31}$$

and

$$\left| \sum_{n > n_0(V)}^{+\infty} \left[\left\{ L_n\left(\frac{|h_k|^2}{2V}\right) - 1 \right\} e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\,n]} \right] \sum_{n=0}^{+\infty} \left[e^{-\beta[(\varepsilon_k - \mu - \frac{g_k}{2V})\,n]} \left(\frac{\beta n^2}{2V}\right) \right] \right|$$
$$\leqslant D_2(k)\, \frac{\beta}{V^{3/4}}\, e^{-\beta(\varepsilon_k - \mu)\, n_0(V)}\, \partial_y^2 f(y = \beta(\varepsilon_k - \mu)). \tag{3.32}$$

Since $\mu < 0$, then for large V the estimates (3.31) and (3.32) are of the order $O(e^{-\beta(\varepsilon_k - \mu)\, V^{(1-\delta)}})$ for some $0 < \delta < 1$. Taking into account (3.26) and (3.27) one concludes that $\Delta_k(V)$, and consequently $(\Gamma_k - \gamma_k)$, have for large V the order $O(e^{-\beta\varepsilon_k} V^{-2})$. This implies that for any $h \in \mathscr{D}$

$$\lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \Gamma_k = \lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \gamma_k (1 + O(e^{-\beta\varepsilon_k} V^{-2})) = \lim_{\Lambda} \prod_{k \in \Lambda^* \setminus \{0\}} \gamma_k, \tag{3.33}$$

which, by (3.20), proves the assertion (3.7). ▮

Remark 3.2. The conditions $0 \leqslant g_- \leqslant g_k \leqslant g_+$ for $k \in \Lambda^* \setminus \{0\}$ can be relaxed. If $0 \leqslant g_k = g_k(V) \leqslant \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \setminus \{0\}$, with $\alpha_k \leqslant \alpha_+ < 1$ and $0 \leqslant \gamma_k \leqslant \gamma_+$, then (3.7) still holds. The proof is obtained by following the same line of reasoning as in the proof of Theorem 3.1.

Remark 3.3. The first result of the Theorem 3.1; i.e. (3.6), is similar to the result for the PBG at densities $\rho \geqslant \rho_c^P(\beta)$ (1.6) in the *canonical* ensemble (β, ρ) (see refs. 22 and 23):

$$\lim_{\Lambda} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{T_\Lambda} (\beta, \rho) \equiv \lim_{\Lambda} \frac{\mathrm{Tr}_{\mathscr{H}_B^{(n)}} (\{ e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} e^{-\beta T_\Lambda} \}^{(n)})}{\mathrm{Tr}_{\mathscr{H}_B^{(n)}} (\{ e^{-\beta T_\Lambda} \}^{(n)})}$$
$$= J_0(\sqrt{2\rho_0^P(\beta, \rho)}\, |\hat{h}_0|),$$

where $n = [V\rho]$ is the integer part of $V\rho$ and $\mathscr{H}_B^{(n)}$ is defined by (1.4). Here

$$\rho_0^P(\beta, \rho) = \rho - \rho_c^P(\beta) = \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{T_\Lambda} (\beta, \rho) \equiv \lim_{\Lambda} \frac{1}{V} \frac{\mathrm{Tr}_{\mathscr{H}_B^{(n)}} (\{ a_0^* a_0 e^{-\beta T_\Lambda} \}^{(n)})}{\mathrm{Tr}_{\mathscr{H}_B^{(n)}} (\{ e^{-\beta T_\Lambda} \}^{(n)})}$$

is the *canonical* density of conventional BE condensate in the PBG. It should be stressed, however, that the limit (3.6) is computed in the grand-canonical ensemble (β, μ) for the non-PBG (1.1).

Corollary 3.4. Let $0 \leqslant g_k = g_k(V) \leqslant \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \setminus \{0\}$, with $\alpha_k \leqslant \alpha_+ < 1$ and $0 \leqslant \gamma_k \leqslant \gamma_+$. Then for $\varepsilon_0 \in \mathbb{R}^1$, $\mu < 0$, and $h \in \mathscr{D}$, one has

$$\mathbb{E}^I(\beta, \mu; h) \equiv \lim_{\Lambda} \mathbb{E}_\Lambda^I(\beta, \mu; h)$$
$$= J_0(\sqrt{2\rho_0^I(\beta, \mu)}\, |\hat{h}_0|) \exp\{ -\tfrac{1}{4} \|h\|^2 - \tfrac{1}{2} A_{\beta,\mu}(h, h) \}, \qquad (3.34)$$

where $\rho_0^I(\beta, \mu)$ and $A_{\beta,\mu}(h_1, h_2)$ are respectively the non-conventional Bose condensate density (1.7) and the positive *closable* sesquilinear form (3.8) with domain $\mathscr{D}$.

Proof. See Theorem 3.1 and Remark 3.2. ▮

The last result is obtained for any *fixed* chemical potential $\mu < 0$. It can be shown[8] that for *finite* volume and any density $\rho \geqslant 0$ there is a one-to-one correspondence between ρ and chemical potential $\mu_\Lambda^I(\beta, \rho)$, which is solution of the equation

$$\rho_\Lambda^I(\beta, \mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu) = \rho. \qquad (3.35)$$

From Corollary 3.4 we have an explicit calculation of the grand-canonical generating functional $\mathbb{E}_\Lambda^I(\beta, \mu; h)$ in the thermodynamic limit. However, for a *fixed* total particle density $\rho \geqslant 0$ one has to evaluate the following thermodynamic limit

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) \equiv \lim_{\Lambda} \mathbb{E}_\Lambda^I(\beta, \mu_\Lambda^I(\beta, \rho); h), \qquad h \in \mathscr{D}. \qquad (3.36)$$

This is *not* guaranteed to equal the limit (3.34) with chemical potential $\mu^I(\beta, \rho) \equiv \lim_\Lambda \mu^I_\Lambda(\beta, \rho)$, since the map $\rho \to \mu^I(\beta, \rho)$ fails to be injective:

$$\mu^I(\beta, \rho) = \lim_{\Lambda} \mu^I_\Lambda(\beta, \rho) = \begin{cases} <0 & \text{for } \rho < \rho^I_c(\beta) \\ =0 & \text{for } \rho \geqslant \rho^I_c(\beta), \end{cases} \tag{3.37}$$

because of conventional BE condensation for dimensions $d > 2$. In this case the total particle density $\rho^I(\beta, \mu)$ (1.8) is saturated for $\mu = 0$:[8] there is a *finite* critical density of particles $\rho^I_c(\theta)$, cf. (1.6), (1.9). The question of equality of the limits (3.34) and (3.36) will be considered in the following paragraphs.

In fact it is linked to another question, which concerns the relation between thermodynamic limit of the *canonical* generating functional defined for the total particle density $\rho \geqslant 0$ by

$$\mathbb{E}^I_{\Lambda, can}(\beta, \rho; h) \equiv \frac{\mathrm{Tr}_{\mathscr{H}_B^{(n)}}(\{W^{\mathscr{F}^B_\Lambda}(h)\, e^{-\beta H^I_\Lambda}\}^{(n)})}{\mathrm{Tr}_{\mathscr{H}_B^{(n)}}(\{e^{-\beta H^I_\Lambda}\}^{(n)})}, \qquad h \in \mathscr{D}_\Lambda, \quad n = [V\rho], \tag{3.38}$$

and the *grand-canonical* generating functional $\tilde{\mathbb{E}}^I(\beta, \rho; h)$ defined by (3.36). In other words, in the thermodynamic limit, for the same particle density ρ, the *canonical* ensemble may *not* yield the same equilibrium state as the *grand-canonical* ensemble; i.e., one may have

$$\mathbb{E}^I_{can}(\beta, \rho; h) \equiv \lim_{\Lambda} \mathbb{E}^I_{\Lambda, can}(\beta, \rho; h) \neq \tilde{\mathbb{E}}^I(\beta, \rho; h), \qquad h \in \mathscr{D}.$$

To answer these questions we notice that

$$\mathbb{E}^I_\Lambda(\beta, \mu^I_\Lambda(\beta)(\beta, \rho); h) = \frac{\sum_{n=0}^{+\infty} e^{-\beta V[-\mu^I_\Lambda(\rho)(\frac{n}{V}) + f^I_\Lambda(\beta, \frac{n}{V})]} \mathbb{E}^I_{\Lambda, can}(\beta, \frac{n}{V}; h)}{\sum_{n=0}^{+\infty} e^{-\beta V[-\mu^I_\Lambda(\rho)(\frac{n}{V}) + f^I_\Lambda(\beta, \frac{n}{V})]}}, \tag{3.39}$$

where $f^I_\Lambda(\beta, \rho)$ is the free-energy density associated with the Hamiltonian H^I_Λ (1.1):

$$f^I_\Lambda(\beta, \rho) \equiv -\frac{1}{\beta V} \ln \mathrm{Tr}_{\mathscr{H}_B^{(n)}}(\{e^{-\beta H^I_\Lambda}\}^{(n)}), \qquad \rho \geqslant 0, \quad n = [V\rho]. \tag{3.40}$$

It is known from ref. 8 that:

(a) $\{f^I_\Lambda(\beta, \rho)\}_{\Lambda \subset \mathbb{R}^d}$ is a family of *strictly* convex functions of $\rho \geqslant 0$, and this is also valid for

$$f^I(\beta, \rho) \equiv \lim_{\Lambda} f^I_\Lambda(\beta, \rho), \tag{3.41}$$

the free-energy density $f^I_\Lambda(\beta, \rho)$ (3.40) in the thermodynamic limit, but in the *smaller* domain: $\rho < \rho^I_c(\beta)$;

(b) the convergence (3.41) implies in this domain (Griffiths lemma, see, e.g., ref. 9) the limit:

$$\partial_\rho f^I(\beta, \rho) = \lim_\Lambda \partial_\rho f^I_\Lambda(\beta, \rho) \equiv \mu^I_{can}(\beta, \rho) < 0, \tag{3.42}$$

which coincides with (3.37):

$$\mu^I_{can}(\beta, \rho) = \mu^I(\beta, \rho), \tag{3.43}$$

for $\rho < \rho^I_c(\beta)$, where this function is one-to-one;

(c) the grand-canonical pressure, $p^I(\beta, \mu) \equiv \lim_\Lambda p^I_\Lambda(\beta, \mu)$, is the Legendre transform:

$$p^I(\beta, \mu) = \sup_{\rho \geqslant 0} \{\mu\rho - f^I(\beta, \rho)\} = \{\mu\rho - f^I(\beta, \rho)\}|_{\rho = \rho^I(\beta, \mu)}, \tag{3.44}$$

and

$$f^I(\beta, \rho) = \sup_{\mu \leqslant 0} \{\mu\rho - p^I(\beta, \mu)\} = \{\mu\rho - p^I(\beta, \mu)\}|_{\mu = \mu^I(\beta, \rho)}, \tag{3.45}$$

where $\rho^I(\beta, \mu < 0) < \rho^I_c(\beta)$ is the function inverse to the injection (3.43):

$$\rho^I(\beta, \mu^I(\beta, \rho)) = \rho;$$

(d) for $\rho \geqslant \rho^I_c(\beta)$, see (1.10) and (3.37), the limit (3.41) is equal to

$$f^I(\beta, \rho) = \sup_{\mu \leqslant 0} \{\mu\rho - p^I(\beta, \mu)\} = -p^I(\beta, \mu = 0), \tag{3.46}$$

i.e., the free-energy density is *not* a strictly convex function in this domain; respectively, $p^I(\beta, \mu = 0) = \sup_{\rho \geqslant 0} \{-f^I(\beta, \rho)\} = -f^I(\beta, \rho \geqslant \rho^I_c(\beta))$, which means that the pressure and the free-energy density are always related by the Legendre transform: *weak equivalence* of ensembles.

By virtue of (a)–(c) we can now apply the Laplace *large deviation principle*[28, 29] to calculate the limit of (3.39) in domain $\rho < \rho^I_c(\beta)$:

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) = \mathbb{E}^I(\beta, \mu^I(\beta, \rho); h) = \mathbb{E}^I_{can}(\beta, \rho; h). \tag{3.47}$$

Notice that for $d = 1, 2$ one has $\rho^I_c = +\infty$, see (1.9). Therefore, Corollary 3.4 together with (3.47) imply the following result.

Theorem 3.5. Let $\rho < \rho_c^I(\beta)$. Then

$$\tilde{\mathbb{E}}^I(\beta, \rho < \rho_c^I(\beta); h) = J_0(\sqrt{2\rho_0^I(\beta, \mu^I(\beta, \rho))}\, |\hat{h}_0|) \times \exp\{-\tfrac{1}{4}\|h\|^2 - \tfrac{1}{2} A_{\beta, \mu^I(\beta, \rho)}(h, h)\}, \tag{3.48}$$

for $d \geqslant 1$, $\varepsilon_0 \in \mathbb{R}^1$, $0 \leqslant g_k \leqslant \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \backslash \{0\}$, with $\alpha_k \leqslant \alpha_+ < 1$ and $0 \leqslant \gamma_k \leqslant \gamma_+$ and $h \in \mathscr{D}$. Here $A_{\beta, \mu}(h_1, h_2)$ and $\mu^I(\beta, \rho < \rho_c^I(\beta)) < 0$ are defined respectively by (3.8) and (3.37).

Consequently, the equality (3.47) shows the *strong equivalence* between the canonical ensemble (β, ρ) and the grand-canonical ensemble (β, μ), for the gauge invariant states, when $\rho < \rho_c^I(\beta)$ (i.e., $\mu < 0$): in the H_Λ^I model (1.1) for a fixed total particle density $\rho < \rho_c^I(\beta)$ the Gibbs state in the grand-canonical ensemble coincides with the one in the canonical ensemble.

However, contrary to the *non-conventional* Bose condensation (1.7) the *conventional* BE condensation $\tilde{\rho}_0^I(\beta, \rho) > 0$ (1.10) violates this strong equivalence. Indeed, by virtue of (d), see (3.46), the limiting measure (3.39) relating two generating functionals is *not degenerate* in domain $\rho \geqslant \rho_c^I(\beta)$. Therefore, similar to the PBG,[22–25] the existence of the critical density $\rho_c^I(\beta)$ implies that for $\rho > \rho_c^I(\beta)$ one has

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) \neq \mathbb{E}_{can}^I(\beta, \rho; h). \tag{3.49}$$

In the next Section 4 we show that, in contrast to (3.47), for $\rho > \rho_c^I(\beta)$ one also gets

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) \neq \mathbb{E}^I(\beta, 0; h). \tag{3.50}$$

4. GIBBS STATES AND CONVENTIONAL CONDENSATION OF TYPE III

Since $\rho_c^I(\beta) < +\infty$ (1.9) only for $d > 2$, we consider $\Lambda \subset \mathbb{R}^{d>2}$. In the interest of simplicity we restrict ourselves to a cubic box of the volume $V = |\Lambda| = L^d$, and we put $g_k = g \geqslant 0$, $k \in \Lambda^*$. Notice that our reasoning in the proofs of Theorems 3.1 and 3.5 used that $\mu < 0$, and that $\mu_\Lambda^I(\beta, \rho < \rho_c^I(\beta)) < 0$, for large V. For $\rho \geqslant \rho_c^I(\beta)$ this is not the case, see Appendix A. This difference modifies essentially the calculations of the thermodynamic limit of the generating functional.

1. Our first step is to refine, for $V \to +\infty$, the asymptotics of the chemical potential $\mu_\Lambda^I(\beta, \rho)$, which is solution of Eq. (3.35):

$$\rho_\Lambda^I(\beta, \mu_\Lambda^I(\beta, \rho)) = \frac{1}{V} \sum_{k \in \Lambda^* \backslash \{0\}} \langle N_k \rangle_{H_\Lambda^I}(\beta, \mu_\Lambda^I(\beta, \rho)) + \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^I}(\beta, \mu_\Lambda^I(\beta, \rho)) = \rho.$$

For a *strictly* positive $g > 0$ this is done in Appendix A, see Theorem A.7:

$$\mu_\Lambda^I(\beta, \rho) = \left(\frac{\rho - \rho_c^I(\beta)}{CV}\right)^{2/(d+2)} + O\left(\frac{1}{V}\right), \tag{4.1}$$

where the constant $C = (2m/\hbar^2)^{d/2}/[g\, 2^{d-2}\pi^{d/2} d(d+2)\, \Gamma(d/2)] > 0$, and $\Gamma(z)$ is the Euler gamma function. Hence, the chemical potential $\mu_\Lambda^I(\beta, \rho \geqslant \rho_c^I(\beta))$, is *non-negative* for large V. By virtue of (3.13) and (3.17) this observation motivates to represent the generating functional (3.5) in the form:

$$\begin{aligned}
\mathbb{E}_\Lambda^I(\beta, \mu_\Lambda^I(\beta, \rho); h) = {} & \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{H_0^I}(\beta, \mu_\Lambda^I(\beta, \rho)) \\
& \times \prod_{k \in D_+^{(\Lambda)}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_k^I}(\beta, \mu_\Lambda^I(\beta, \rho)) \\
& \times \prod_{k \in D_-^{(\Lambda)}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_k^I}(\beta, \mu_\Lambda^I(\beta, \rho)),
\end{aligned} \tag{4.2}$$

with

$$\begin{aligned}
D_-^{(\Lambda)} &\equiv \left\{ k \in \Lambda^* \backslash \{0\} : \varepsilon_k - \mu_\Lambda^I(\beta, \rho) - \frac{g_k}{2V} < 0 \right\}, \\
D_+^{(\Lambda)} &\equiv \left\{ k \in \Lambda^* \backslash \{0\} : \varepsilon_k - \mu_\Lambda^I(\beta, \rho) - \frac{g_k}{2V} \geqslant 0 \right\}.
\end{aligned} \tag{4.3}$$

Remark 4.1. It is shown in Appendix A, that for $\rho > \rho_c^I(\beta)$ and $g_k \geqslant g > 0$ the non-extensive condensation $\tilde{\rho}_0^I(\beta, \rho)$ (1.10) is concentrated on the set $D_-^{(\Lambda)}$:

$$\tilde{\rho}_0^I(\beta, \rho) = \lim_\Lambda \frac{1}{V} \sum_{k \in D_-^{(\Lambda)}} \langle N_k \rangle_{H_\Lambda^I}(\beta, \mu_\Lambda^I(\beta, \rho)) = \rho - \rho_c^I(\beta) > 0, \tag{4.4}$$

see Lemma 1.6 in Appendix A.

2. Since in the proof of Theorem 3.1(i) the sign of μ is irrelevant, the same line of reasoning gives that for $\mu = \mu_\Lambda^I(\beta, \rho \geqslant \rho_c^I(\beta))$, (4.1), and for any $h \in \mathscr{D}$, $\varepsilon_0 \in \mathbb{R}^1$:

$$\lim_\Lambda \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{H_0^I}(\beta, \mu_\Lambda^I(\beta, \rho)) = J_0(\sqrt{2\rho_0^I(\beta, 0)}\, |\hat{h}_0|). \tag{4.5}$$

Here the non-conventional Bose-condensate density $\rho_0^I(\theta, 0)$ is defined for $\mu = 0$ by (1.7).

3. Now, using the asymptotics (4.1) we can evaluate the thermodynamic limit of

$$\prod_{k\in D_+^{(\Lambda)}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k}a_k+h_k a_k^*)}\rangle_{H_k^I}(\beta,\mu_\Lambda^I(\beta,\rho)), \tag{4.6}$$

see (4.2). Indeed, by inspection of the line of reasoning (3.17)–(3.32) we find that it is only the inequality $\varepsilon_k-\mu_\Lambda^I(\beta,\rho)-\frac{g_k}{2V}>0$ which one needs that the limit (3.33) be valid. Therefore, taking into account (4.1) we get

$$\lim_\Lambda \prod_{k\in D_+^{(\Lambda)}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k}a_k+h_k a_k^*)}\rangle_{H_k^I}(\beta,\mu_\Lambda^I(\beta,\rho))=\exp\{-\tfrac{1}{4}\|h\|^2-\tfrac{1}{2}A_{\beta,0}(h,h)\}, \tag{4.7}$$

for $h\in\mathscr{D}$ and $\rho\geqslant\rho_c^I(\theta)$, with the sesquilinear form $A_{\beta,\mu=0}(h_1,h_2)$ defined by (3.8).

4. Finally, using the asymptotics (4.1) we have to compute the thermodynamic limit of the last factor in (4.2):

$$\lim_\Lambda \prod_{k\in D_-^{(\Lambda)}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k}a_k+h_k a_k^*)}\rangle_{H_k^I}(\beta,\mu_\Lambda^I(\beta,\rho)). \tag{4.8}$$

Because of *non-negative* $\mu_\Lambda^I(\beta,\rho)$, and of the *non-extensive* BE condensation $\tilde{\rho}_0^I(\theta,\rho)$ (4.4), that spreads over the modes $k\in D_-^{(\Lambda)}$ (Remark 4.1), this calculation is a more subtle matter than the thermodynamic limit of (4.6). In particular, the exact knowledge of the asymptotics (4.1) of $\mu_\Lambda^I(\beta,\rho)$ becomes essential to find the limit of (4.8).

By (3.17) we get:

$$\Gamma_k=\langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k}a_k+h_k a_k^*)}\rangle_{H_k^I}=e^{-\frac{1}{4V}|h_k|^2}\sum_{n=0}^{+\infty}\nu_{\Lambda,k}(\beta,\rho;n)\,L_n\left(\frac{|h_k|^2}{2V}\right), \tag{4.9}$$

where

$$\nu_{\Lambda,k}(\beta,\rho;n)\equiv\frac{e^{-\beta[(\varepsilon_k-\mu_\Lambda^I(\beta,\rho)-\frac{g_k}{2V})\,n+\frac{g_k}{2V}n^2]}}{\sum_{n=0}^{+\infty}e^{-\beta[(\varepsilon_k-\mu_\Lambda^I(\beta,\rho)-\frac{g_k}{2V})\,n+\frac{g_k}{2V}n^2]}} \tag{4.10}$$

is the family of probability measures $\{\nu_{\Lambda,k}(\beta,\rho;n)\}_{\Lambda\subset\mathbb{R}^d,\,k\in\Lambda^*}$, cf. (3.14) and (A17). Consequently

$$\ln\prod_{k\in D_-^{(\Lambda)}}\Gamma_k=\left(-\frac{1}{4V}\right)\sum_{k\in D_-^{(\Lambda)}}|h_k|^2+\sum_{k\in D_-^{(\Lambda)}}\ln\sum_{n=0}^{+\infty}\nu_{\Lambda,k}(\beta,\rho;n)\,L_n\left(\frac{|h_k|^2}{2V}\right). \tag{4.11}$$

Since $\mu_\Lambda^I(\beta, \rho \geqslant \rho_c^I(\beta)) \to 0$, the thermodynamic limit of the first term in (4.11) is

$$\lim_{\Lambda} \left(-\frac{1}{4V}\right) \sum_{k \in D_-^{(\Lambda)}} |h_k|^2 = \lim_{\delta \to 0} \left(-\frac{1}{4}\right) \frac{1}{(2\pi)^d} \int_{\{k:\, \varepsilon_k \leqslant \delta\}} d^d k \, |\hat{h}_k|^2 = 0, \tag{4.12}$$

for any $h \in \mathscr{D}$. By virtue of the asymptotics (4.1) and by definition of domain $D_-^{(\Lambda)}$ we get for the limit of the second term:

$$\lim_{\Lambda} \sum_{k \in D_-^{(\Lambda)}} \ln \sum_{n=0}^{+\infty} \nu_{\Lambda, k}(\beta, \rho; n) \, L_n \left(\frac{|h_k|^2}{2V}\right)$$

$$= \lim_{\Lambda} \sum_{s \in \mathscr{S}_B} \ln \sum_{n=0}^{+\infty} \nu_{\Lambda, k(s)}(V^{1-2\gamma}\beta, \rho; n/V^{1-\gamma}) \, L_n \left(\frac{|h_{k(s)/V^{\gamma/2}}|^2}{2V}\right). \tag{4.13}$$

Here $\gamma = 2/(d+2)$, the set:

$$\mathscr{S}_B \equiv \left\{ s = \{s_\alpha\}_{\alpha=1}^d \in \mathbb{Z}^d \setminus \{0\} : \frac{\hbar^2}{2m} (2\pi)^2 \sum_{\alpha=1}^d (s_\alpha / V^{1/d - \gamma/2})^2 \leqslant B \right\}, \tag{4.14}$$

and we put $k(s) \equiv 2\pi s / V^{1/d-\gamma/2}$. Since for $d > 2$ one has $1 - 2\gamma > 0$, the family of the *scaled* probability measures:

$$\{\nu_{\Lambda, k(s)}(V^{1-2\gamma}\beta, \rho; n/V^{1-\gamma})\}_{\Lambda \subset \mathbb{R}^d,\, s \in \mathscr{S}_{\mathscr{B}}}, \tag{4.15}$$

see (4.10), verifies the Laplace *large deviation principle*[28, 29] with the support at the point $\lim_\Lambda \bar{n}(V)/V^{1-\gamma} = (B - \varepsilon_k(s))/g$, cf. Theorem A.7. This remark, together with the asymptotics (3.30) of the Laguerre polynomials and the continuity of $h \in \mathscr{D}$, gives:

$$\lim_{\Lambda} \sum_{s \in \mathscr{S}_B} \ln \sum_{n=0}^{+\infty} \nu_{\Lambda, k(s)}(V^{1-2\gamma}\beta, \rho; n/V^{1-\gamma}) \, L_n \left(\frac{|h_{k(s)/V^{\gamma/2}}|^2}{2V}\right)$$

$$= \lim_{\Lambda} \sum_{s \in \mathscr{S}_B} \ln L_{\bar{n}(V) = V^{1-\gamma}(B - \varepsilon_k(s))/g} \left(\frac{|h_{k(s)/V^{\gamma/2}}|^2}{2V}\right)$$

$$= -\lim_{\Lambda} \sum_{s \in \mathscr{S}_B} V^{1-\gamma} \frac{(B - \varepsilon_{k(s)})}{g} \frac{|h_{k(s)/V^{\gamma/2}}|^2}{2V} = -|\hat{h}_0|^2 \frac{1}{(2\pi)^d} \int_{\{k:\, \varepsilon_k \leqslant B\}} d^d k \, \frac{B - \varepsilon_k}{g}$$

$$= -\frac{1}{2} |\hat{h}_0|^2 (\rho - \rho_c^I(\beta)). \tag{4.16}$$

Here we used that $\gamma = 1 - d\gamma/2$ to obtain in the limit the integral, and (A.25) to get the last equality. Taking (4.12) and (4.16) into account we finally get for (4.8):

$$\lim_{\Lambda} \prod_{k \in D_{-}^{(\Lambda)}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{H_k^I} (\beta, \mu_\Lambda^I(\beta, \rho)) = \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 (\rho - \rho_c^I(\beta))\}$$
$$= \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 \tilde{\rho}_0^I(\beta, \rho)\}, \quad (4.17)$$

where $\tilde{\rho}_0^I(\beta, \rho)$ is density of the *type III* BE condensation, see (1.10).

The results of 1–4 can be summed up as follows:

Theorem 4.2. Let $\rho \geqslant \rho_c^I(\beta)$. Then

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) \equiv \lim_{\Lambda} \mathbb{E}_\Lambda^I(\beta, \mu_\Lambda^I(\beta, \rho); h)$$
$$= J_0(\sqrt{2\rho_0^I(\beta, 0)}\, |\hat{h}_0|) \exp\{-\tfrac{1}{4} \|h\|^2 - \tfrac{1}{2} A_{\beta, 0}(h, h)\}$$
$$\times \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 (\rho - \rho_c^I(\beta))\}, \quad (4.18)$$

for $d > 2$, $\varepsilon_0 \in \mathbb{R}^1$, $g_k = g > 0$ and $h \in \mathscr{D}$. Here $A_{\beta, \mu}(h_1, h_2)$ and $\mu_\Lambda^I(\beta, \rho \geqslant \rho_c^I(\beta)) \geqslant 0$ are defined respectively by (3.8) and (4.1).

Corollary 4.3. Comparing (3.34) and (4.18) one gets that the presence of the *conventional* BE condensation $\tilde{\rho}_0^I(\beta, \rho) > 0$, see (1.10), (4.4), implies:

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) = \mathbb{E}^I(\beta, 0; h) \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 \tilde{\rho}_0^I(\beta, \rho)\}, \quad (4.19)$$

cf. (3.50). This means that at the point $\mu = \lim_\Lambda \mu_\Lambda^I(\beta, \rho \geqslant \rho_c^I(\beta)) = 0$ the grand-canonical equilibrium state is not unique. There is a family of states enumerated by the BE condensate density $\tilde{\rho}_0^I(\beta, \rho) = \rho - \rho_c^I(\beta)$.

This effect is well-known in the PBG:[23–25]

Proposition 4.4. For isotropic dilation of a rectangular container $\Lambda \subset \mathbb{R}^{d>2}$ the grand-canonical generating functional

$$\tilde{\mathbb{E}}^P(\beta, \rho; h) \equiv \lim_{\Lambda} \frac{\mathrm{Tr}_{\mathscr{F}_\Lambda^B}(\prod_{k \in \Lambda^*} e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} e^{-\beta(T_\Lambda - \mu_\Lambda^P(\beta, \rho) N_\Lambda)})}{\mathrm{Tr}_{\mathscr{F}_\Lambda^B}(e^{-\beta(T_\Lambda - \mu_\Lambda^P(\beta, \rho) N_\Lambda)})}$$
$$= \lim_{\Lambda} \exp\{-\tfrac{1}{4} \|h\|^2 - \tfrac{1}{2} A_{\beta, \mu_\Lambda^P(\beta, \rho)}(h, h)\}$$
$$\times \lim_{\Lambda} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{T_\Lambda} (\beta, \mu_\Lambda^P(\beta, \rho))$$
$$= \exp\{-\tfrac{1}{4} \|h\|^2 - \tfrac{1}{2} A_{\beta, \mu^P(\beta, \rho)}(h, h)\} \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 \rho_0^P(\beta, \rho)\}. \quad (4.20)$$

Here $\mu_\Lambda^P(\beta, \rho) < 0$ is solution of the equation

$$\rho_\Lambda^P(\beta, \mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{T_\Lambda} (\beta, \mu) = \frac{1}{V} \frac{\mathrm{Tr}_{\mathscr{F}_\Lambda^B}(N_\Lambda e^{-\beta(T_\Lambda - \mu N_\Lambda)})}{\mathrm{Tr}_{\mathscr{F}_\Lambda^B}(e^{-\beta(T_\Lambda - \mu N_\Lambda)})} = \rho.$$

Similar to (3.37) and (4.4), one has:

$$\mu^P(\beta, \rho) = \lim_\Lambda \mu_\Lambda^P(\beta, \rho) = \begin{cases} < 0 & \text{for} \quad \rho < \rho_c^P(\beta) \\ = 0 & \text{for} \quad \rho \geqslant \rho_c^P(\beta), \end{cases}$$

and the type I conventional BE condensation in the single mode $k = 0$:

$$\tilde{\rho}_0^P(\beta, \rho) = \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{T_\Lambda} (\beta, \mu_\Lambda^P(\beta, \rho)) = \begin{cases} = 0 & \text{for} \quad \rho \leqslant \rho_c^P(\beta) \\ = \rho - \rho_c^P(\beta) & \text{for} \quad \rho > \rho_c^P(\beta). \end{cases} \tag{4.21}$$

Hence, similar to (4.19), one has

$$\tilde{\mathbb{E}}^P(\beta, \rho > \rho_c^P(\beta); h) = \mathbb{E}^P(\beta, 0; h) \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 \tilde{\rho}_0^P(\beta, \rho)\}.$$

Moreover, the remarks (a)–(d) are valid for PBG. For $\rho \leqslant \rho_c^P(\beta)$ there is a *strong equivalence* of ensembles expressed by:

$$\tilde{\mathbb{E}}^P(\beta, \rho; h) = \mathbb{E}^P(\beta, \mu^P(\beta, \rho); h) = \mathbb{E}_{can}^P(\beta, \rho; h), \tag{4.22}$$

cf. (3.47). Whereas for $\rho > \rho_c^P(\beta)$, the functional $\mathbb{E}_{can}^P(\beta, \rho; h)$ and the *non-degenerate* measure in (3.39), relating the canonical and the grand-canonical generating functionals, can be calculated explicitly.[23–25] This gives:

$$\tilde{\mathbb{E}}^P(\beta, \rho; h) = \int_{\mathbb{R}^1} K_{\rho, \rho_c^P(\beta)}(dx)\, \mathbb{E}_{can}^P(\beta, x; h). \tag{4.23}$$

Here

$$\mathbb{E}_{can}^P(\beta, x > \rho_c^P(\beta); h) = \exp\{-\tfrac{1}{4} \|h\|^2 - \tfrac{1}{2} A_{\beta,0}(h, h)\}\, J_0(\sqrt{2(x - \rho_c^P(\beta))}\, |\hat{h}_0|), \tag{4.24}$$

and

$$K_{\rho, \rho_c^P(\beta)}(dx) = \begin{cases} 0 & \text{for} \quad x \leqslant \rho_c^P(\beta) \\ \{\rho - \rho_c^P(\beta)\}^{-1} \exp\left\{ -\dfrac{x - \rho_c^P(\beta)}{\rho - \rho_c^P(\beta)} \right\} dx & \text{for} \quad x > \rho_c^P(\beta), \end{cases} \tag{4.25}$$

is known as the *Kac measure* for $\rho > \rho_c^P(\beta)$. Since by (a)–(c) the limiting measure in (3.39) is *degenerate*:

$$K_{\rho,\rho_c^P(\beta)}(dx) = \delta(x-\rho)\,dx, \tag{4.26}$$

for $\rho \leqslant \rho_c^P(\beta)$, cf. (4.22), the representation (4.23) is valid for any $\rho \geqslant 0$.

5. CONCLUDING REMARKS

5.1. To answer the question about the rôle of the *type* III BE condensation in determining the generating functional (4.19), let us consider instead of (1.1) a truncated model with the Hamiltonian H_Λ^0.

Since Theorem 3.1 is valid for $g_+ = 0$ (or $\gamma_+ = 0$), the generating functional for the model H_Λ^0 has for $\mu < 0$ the same form:

$$\begin{aligned}\mathbb{E}^0(\beta,\mu;h) &= \lim_\Lambda \prod_{k\in\Lambda^*} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k}a_k + h_k a_k^*)}\rangle_{H_\Lambda^0}(\beta,\mu)\\ &= J_0(\sqrt{2\rho_0^0(\beta,\mu)}\,|\hat{h}_0|)\exp\{-\tfrac{1}{4}\|h\|^2 - \tfrac{1}{2}A_{\beta,\mu}(h,h)\},\end{aligned} \tag{5.1}$$

as for H_Λ^I, cf. (3.34). Here $\rho_0^0(\beta,\mu) = \rho_0^I(\beta,\mu)$, see (1.7). Again, in this domain we have the *strong equivalence* of ensembles (3.47):

$$\tilde{\mathbb{E}}^0(\beta,\rho;h) = \mathbb{E}^0(\beta,\mu^0(\beta,\rho);h) = \mathbb{E}^0_{can}(\beta,\rho;h), \tag{5.2}$$

where $\mu^0(\beta,\rho) = \mu^I(\beta,\rho)$ and $\rho_c^0(\beta) = \rho_c^I(\beta)$, see (3.37).

On the other hand, for $\rho > \rho_c^0(\beta)$ the model H_Λ^0 manifests (instead of the *type* III) the BE condensation of the *type* I, see ref. 8. More precisely, for dilatation of a *cubic* Λ there is the BE condensation in $2d$ modes:

$$\mathscr{K}_{2d} \equiv \{k\in\Lambda^* : \{k_i^\alpha\}_{i=1}^d = (0,0,\ldots,0,k_\alpha = \pm 2\pi/V^{1/d},0,\ldots,0),\ \alpha = 1,2,\ldots,d\},$$

such that

$$\lim_\Lambda \frac{1}{V}\langle N_{k\in\mathscr{K}_{2d}}\rangle_{H_\Lambda^0}(\beta,\mu_\Lambda^0(\beta,\rho>\rho_c^0(\beta))) = \frac{1}{2d}(\rho-\rho_c^0(\beta)), \tag{5.3}$$

and

$$\lim_\Lambda \frac{1}{V}\langle N_{k\in\Lambda^*\setminus\{\mathscr{K}_{2d}\cup\{0\}\}}\rangle_{H_\Lambda^0}(\beta,\mu_\Lambda^0(\beta,\rho>\rho_c^0(\beta))) = 0. \tag{5.4}$$

It corresponds to the asymptotics:

$$\mu_\Lambda^0(\beta, \rho > \rho_c^0(\beta)) = \varepsilon_{k_\alpha} - \frac{2d}{V\beta(\rho - \rho_c^0(\beta))} + O\left(\frac{1}{V}\right) \tag{5.5}$$

of the solution of the equation

$$\rho = \frac{1}{V} \sum_{k \in \Lambda^*} \langle N_k \rangle_{H_\Lambda^0} (\beta, \mu_\Lambda^0(\beta, \rho)). \tag{5.6}$$

Since $\varepsilon_k - \mu_\Lambda^0(\beta, \rho) > 0$ for $k \neq 0$, following the same line of reasoning as in calculation of (4.2) we get for

$$\begin{aligned} \mathbb{E}_\Lambda^0(\beta, \mu_\Lambda^0(\beta, \rho); h) = {} & \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_0} a_0 + h_0 a_0^*)} \rangle_{U_\Lambda^0} (\beta, \mu_\Lambda^0(\beta, \rho)) \\ & \times \prod_{k \in \mathscr{K}_{2d}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{T_\Lambda} (\beta, \mu_\Lambda^0(\beta, \rho)) \\ & \times \prod_{k \in \Lambda^* \setminus \{\mathscr{K}_{2d} \cup \{0\}\}} \langle e^{\frac{i}{\sqrt{2V}}(\overline{h_k} a_k + h_k a_k^*)} \rangle_{T_\Lambda} (\beta, \mu_\Lambda^0(\beta, \rho)) \end{aligned} \tag{5.7}$$

the limit:

$$\begin{aligned} \tilde{\mathbb{E}}^0(\beta, \rho; h) & \equiv \lim_\Lambda \mathbb{E}_\Lambda^0(\beta, \mu_\Lambda^0(\beta, \rho); h) \\ & = J_0(\sqrt{2\rho_0^0(\beta, 0)}\, |\hat{h}_0|) \exp\{-\tfrac{1}{4} \|h\|^2 - \tfrac{1}{2} A_{\beta, 0}(h, h)\} \\ & \quad \times \exp\{-\tfrac{1}{2} |\hat{h}_0|^2 (\rho - \rho_c^0(\beta))\}, \end{aligned} \tag{5.8}$$

which *coincides* with the generating functional (4.18), or (4.19).

Theorem 5.1. The generating functionals for the models H_Λ^0 and H_Λ^I are identical:

$$\tilde{\mathbb{E}}^0(\beta, \rho; h) = \tilde{\mathbb{E}}^I(\beta, \rho; h), \tag{5.9}$$

for any β and ρ.

5.2. Notice that the relation between grand-canonical and canonical generating functionals (4.23) for the PBG can be seen by means of the identity:

$$\exp\left(-\frac{1}{2}\lambda z^2\right) = \int_0^{+\infty} \frac{dt}{\lambda}\, e^{-t/\lambda} J_0(\sqrt{2t}\, |z|), \tag{5.10}$$

for $\lambda > 0$. It yields the representation of the grand-canonical generating functionals (4.19) (or (5.8)) via the *Kac measure*:

$$\tilde{\mathbb{E}}^I(\beta, \rho; h) = \int_{\mathbb{R}^1} K_{\rho, \rho_c^I(\beta)}(dx)\, \mathbb{E}^I(\beta, \mu = 0; h)\, J_0(\sqrt{2(x - \rho_c^I(\beta))}\, |\hat{h}_0|). \quad (5.11)$$

Following (4.23) and (4.24) this gives a temptation to identify

$$\mathbb{E}^I(\beta, \mu = 0; h)\, J_0(\sqrt{2(\rho - \rho_c^I(\beta))}\, |\hat{h}_0|) = \mathbb{E}^I_{can}(\beta, \rho; h), \quad (5.12)$$

for $\rho > \rho_c^I(\beta)$, with the generating functional $\mathbb{E}^I_{can}(\beta, \rho; h)$ of the model H^I_Λ (or H^0_Λ) in the *canonical* ensemble. But actually we know neither $\mathbb{E}^I_{can}(\beta, \rho; h)$, nor $\mathbb{E}^0_{can}(\beta, \rho; h)$, for $\rho > \rho_c^I(\beta)$, so we are unable to check this hypothesis.

For $\rho \leqslant \rho_c^I(\beta)$ the representation of the grand-canonical generating functional via the degenerate *Kac measure* (4.26) and the canonical generating functional follows directly from (3.47), or (5.2).

5.3. In the present paper we are not concerned with the explicit form of the CCR representations corresponding to the generating functional $\tilde{\mathbb{E}}^I(\beta, \rho; h)$ for different densities ρ. Therefore, we limit ourselves only by few remarks:

– In domain $\rho \leqslant \rho(\beta, \varepsilon_0)$ one has:

$$\tilde{\mathbb{E}}^I(\beta, \rho < \rho_c^I(\beta); h) = \exp\{-\tfrac{1}{4}\, \|h\|^2 - \tfrac{1}{2}\, A_{\beta, \mu^I(\beta, \rho) < 0}(h, h)\}, \quad (5.13)$$

where quadratic form $\frac{1}{4}\, \|h\|^2 + \frac{1}{2}\, A_{\beta, \mu^I(\beta, \rho)}(h, h)$ is *closable*. It is known that in this case the representation is a *factor* corresponding to the class of *quasi-free* states, see refs. 30 and 27 for details.

– By virtue of (5.10) the $\tilde{\mathbb{E}}^I(\beta, \rho; h)$ in domain $\rho(\beta, \varepsilon_0) < \rho < \rho_c^I(\beta)$ is the (inverse) Laplace transform of a generating functional with the *non-closable* quadratic form $\frac{1}{2}\, \lambda\, |\hat{h}_0|^2 + \frac{1}{4}\, \|h\|^2 + \frac{1}{2}\, A_{\beta, \mu^I(\beta, \rho)}(h, h)$. Hence, the representation is *not* a factor.

– Because of the term $-\frac{1}{2}\, |\hat{h}_0|^2\, \tilde{\rho}_0^I(\beta, \rho)$, see (4.19), the same remark is valid for representation in domain $\rho \geqslant \rho_c^I(\beta)$. For details of construction of the representation in these cases see ref. 24.

APPENDIX A

The aim of this appendix is to investigate the *type III* BE condensate in the model (1.1), and to add some new results to what is known since.[8, 9]

The essential point is to obtain the asymptotics of $\mu^I_\Lambda(\rho)$, which is the solution the of equation

$$\rho = \rho^I_\Lambda(\beta, \mu^I_\Lambda(\beta, \rho)) = \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \langle N_k \rangle_{H^I_\Lambda} (\beta, \mu^I_\Lambda(\beta, \rho)) + \left\langle \frac{N_0}{V} \right\rangle_{H^I_\Lambda} (\beta, \mu^I_\Lambda(\beta, \rho)), \tag{A.1}$$

see (3.35). We recall that by (1.7) one has

$$\lim_\Lambda \left\langle \frac{N_0}{V} \right\rangle_{H^I_\Lambda} (\beta, \mu^I_\Lambda(\beta, \rho)) = \rho^I_0(\theta, \mu^I(\beta, \rho)),$$

where we put $\mu^I(\beta, \rho) \equiv \lim_\Lambda \mu^I_\Lambda(\beta, \rho)$. To simplify the arguments, we consider below a *cubic* box $\Lambda \subset \mathbb{R}^{d>2}$ of the volume $V = |\Lambda| = L^d$, and (at the very last moment) we put all constants $\{g_k\}_{k \in \Lambda^*}$ to be equal to $g > 0$.

A.1. Let $\tilde{D}^{(\Lambda)}_-$ and $\tilde{D}^{(\Lambda)}_+$ be two sets defined by

$$\begin{aligned} \tilde{D}^{(\Lambda)}_- &\equiv \left\{ k \in \Lambda^* \setminus \{0\} : \varepsilon_k - \mu^I_\Lambda(\beta, \rho) - \frac{g_k}{V} \leqslant 0 \right\} \\ \tilde{D}^{(\Lambda)}_+ &\equiv \left\{ k \in \Lambda^* \setminus \{0\} : \varepsilon_k - \mu^I_\Lambda(\beta, \rho) - \frac{g_k}{V} > 0 \right\}, \end{aligned} \tag{A.2}$$

cf. (4.3). Then (A.1) transforms into

$$\rho = \frac{1}{V} \sum_{k \in \tilde{D}^{(\Lambda)}_-} \langle N_k \rangle_{H^I_\Lambda} (\beta, \mu^I_\Lambda(\beta, \rho)) + \frac{1}{V} \sum_{k \in \tilde{D}^{(\Lambda)}_+} \langle N_k \rangle_{H^I_\Lambda} (\beta, \mu^I_\Lambda(\beta, \rho)) + \left\langle \frac{N_0}{V} \right\rangle_{H^I_\Lambda} (\beta, \mu^I_\Lambda(\beta, \rho)). \tag{A.3}$$

Notice that, since $\varepsilon_{k \neq 0} = O(V^{-2/d})$ for $d > 2$, the set $\tilde{D}^{(\Lambda)}_- = \varnothing$, if $\mu^I_\Lambda(\beta, \rho) \leqslant 0$ for large V.

It is natural to anticipate that, due to the repulsive interaction, the occupation number of the mode k decreases in comparison to a perfect Bose gas at a modified chemical potential. The following lemma provides a bound of this type sufficiently good for our purposes for $k \in \tilde{D}^{(\Lambda)}_+$. the bound is meaningless for $k \notin \tilde{D}^{(\Lambda)}_+$

Lemma A.1 (Ref. 8). Let $g_+ \geqslant g_k \geqslant g_- > 0$ for $k \in \Lambda^* \setminus \{0\}$. Then for any $\varepsilon_0 \in \mathbb{R}^1$ and $k \in \tilde{D}_+^{(\Lambda)}$, one has the estimate:

$$\langle N_k \rangle_{H_\Lambda^I} \leqslant \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda^I(\beta,\rho) - \frac{g_k}{V})} - 1}, \tag{A.4}$$

for V sufficiently large.

Proof. By the correlation inequalities for the Gibbs state $\omega_\Lambda^I(-) \equiv \langle - \rangle_{H_\Lambda^I}$ (see refs. 31 and 32):

$$\beta \omega_\Lambda^I(X^*[H_\Lambda^I(\mu), X]) \geqslant \omega_\Lambda^I(X^*X) \ln \frac{\omega_\Lambda^I(X^*X)}{\omega_\Lambda^I(XX^*)}, \tag{A.5}$$

where X is an observable from the domain of the commutator $[H_\Lambda^I(\mu), .]$ with $H_\Lambda^I(\mu) \equiv H_\Lambda^I - \mu N_\Lambda$, we can deduce

$$\beta \omega_\Lambda^I(a_k^*[H_\Lambda^I(\mu), a_k]) \geqslant \omega_\Lambda^I(N_k) \ln \frac{\omega_\Lambda^I(N_k)}{\omega_\Lambda^I(N_k)+1}, \tag{A.6}$$

for $X = a_k$ $(k \neq 0)$. Since for $k \neq 0$,

$$[H_\Lambda^I(\mu), a_k] = (\mu - \varepsilon_k)\, a_k - \frac{g_k}{V}\, a_k^* a_k^2,$$

from (A.6) one finds for $\mu = \mu_\Lambda^I(\rho)$

$$\beta \omega_\Lambda^I\left(\left(\mu_\Lambda^I(\rho) + \frac{g_k}{V} - \varepsilon_k\right) N_k - \frac{g_k}{V}\, N_k^2\right) \geqslant \omega_\Lambda^I(N_k) \ln \frac{\omega_\Lambda^I(N_k)}{\omega_\Lambda^I(N_k)+1},$$

which gives:

$$\beta\left(\varepsilon_k - \mu_\Lambda^I(\rho) - \frac{g_k}{V}\right) \omega_\Lambda^I(N_k) \leqslant \omega_\Lambda^I(N_k) \ln \frac{\omega_\Lambda^I(N_k)+1}{\omega_\Lambda^I(N_k)}. \tag{A.7}$$

Now the rest of the proof is essentially due to solution for $x \geqslant 0$ of the inequality (A.7):

$$b_k \leqslant \ln \frac{x+1}{x}, \tag{A.8}$$

where

$$x \equiv \omega_\Lambda^I(N_k), \qquad b_k \equiv \beta\left(\varepsilon_k - \mu_\Lambda^I(\rho) - \frac{g_k}{V}\right). \tag{A.9}$$

Since for $k \in \tilde{D}_{+}^{(\Lambda)}$ we have $b_k > 0$, the inequality (A.8) implies (A.4) for sufficiently large Λ. ∎

Remark A.2 (Ref. 8). Because of repulsive interaction $g_k \geqslant g_- > 0$ for $k \in \Lambda^*$, see (1.2), the finite-volume pressure $p_\Lambda^I(\beta, \mu)$ of the model (1.1) is a convex function of $\mu \in \mathbb{R}^1$ such that $\lim_\Lambda p_\Lambda^I(\beta, \mu > 0) = +\infty$. This means that (in contrast to the PBG) for large ρ the solution $\mu_\Lambda^I(\rho)$ of Eq. (A.1) should be *positive*. By convexity:

$$\frac{p_\Lambda^I(\beta, \mu) - p_\Lambda^I(\beta, \mu = 0)}{\mu} \leqslant \partial_\mu p_\Lambda^I(\beta, \mu) = \rho_\Lambda^I(\beta, \mu) \tag{A.10}$$

for $\mu > 0$, one gets that $\lim_\Lambda \rho_\Lambda^I(\beta, \mu > 0) = +\infty$. Therefore, $\lim_\Lambda \mu_\Lambda^I(\rho) \leqslant 0$; i.e., the *positive* solution of (A.1) must go to zero in the thermodynamic limit.

Corollary A.3. For $\rho > \rho_c^I(\beta)$ and V sufficiently large the solution $\mu_\Lambda^I(\beta, \rho > \rho_c^I(\beta)) > 0$.

Proof. Suppose that $\mu_\Lambda^I(\beta, \rho) \leqslant 0$ for any ρ and Λ. Then $\tilde{D}_{-}^{(\Lambda)} = \varnothing$; i.e., $\tilde{D}_{+}^{(\Lambda)} = \Lambda^* \setminus \{0\}$, and by Lemma A.1 one gets the estimate:

$$\begin{aligned} &\frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) + \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) \\ &\quad \leqslant \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda^I(\beta, \rho) - \frac{g_k}{V})} - 1} + \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)). \end{aligned}$$

Then, by virtue of $\mu_\Lambda^I(\beta, \rho) \leqslant 0$, we get the following inequalities in the thermodynamic limit:

$$\begin{aligned} \rho &= \lim_\Lambda \frac{1}{V} \sum_{k \in \Lambda^*} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) \\ &\leqslant \lim_\Lambda \rho_\Lambda^P(\beta, \mu_\Lambda^I(\beta, \rho)) + \lim_\Lambda \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) \\ &\leqslant \rho_c^P(\beta) + \rho_0^I(\beta, \mu = 0) = \rho_c^I(\beta), \end{aligned} \tag{A.11}$$

see (1.7)–(1.9). But this is impossible for $\rho > \rho_c^I(\beta)$, that proves the assertion. ∎

Therefore, Remark A.2 and Corollary 1.3 state that

$$\lim_\Lambda \mu_\Lambda^I(\beta, \rho > \rho_c^I(\beta)) = 0. \tag{A.12}$$

Corollary A.4. The representation (A.3), together with arguments of Corollary A.3 and (A.12), allow to refine the localisation of the *non-extensive* BE condensation (1.10):

$$\begin{aligned}\tilde{\rho}_0^I(\beta,\rho) &= \lim_{\Lambda} \frac{1}{V} \sum_{k \in \tilde{D}_-^{(\Lambda)}} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta,\rho)) \\ &= \rho - \lim_{\Lambda} \frac{1}{V} \sum_{k \in \tilde{D}_+^{(\Lambda)}} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta,\rho)) \\ &\quad - \lim_{\Lambda} \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta,\rho)) \\ &= \rho - \rho_c^I(\beta). \end{aligned} \tag{A.13}$$

A.2. Our next step is to calculate the *asymptotics* of $\mu_\Lambda^I(\beta, \rho > \rho_c^I(\beta))$ in (A.12). To this end suppose that for $V \to +\infty$ it has the form:

$$\mu_\Lambda^I(\beta, \rho > \rho_c^I(\beta)) = \frac{B}{V^\gamma} + o(V^{-\gamma}), \tag{A.14}$$

with $\gamma > 0$ and $B > 0$ that should be defined from Eq. (A.1).

Remark A.5. Suppose that $\gamma > 2/d$. Since $\varepsilon_{k \neq 0} = O(V^{-2/d})$, then the set $\tilde{D}_-^{(\Lambda)} = \varnothing$, for large V. Therefore, the same line of reasoning as in Corollary A.3 produces a contradiction to our main assumption: $\rho > \rho_c^I(\beta)$. Hence, we must have:

$$\gamma \leqslant 2/d. \tag{A.15}$$

By virtue of additive structure of the Hamiltonian (3.2), for any $k \in \Lambda^*$ we get that

$$\left\langle \frac{N_k}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta,\rho)) = \left\langle \frac{N_k}{V} \right\rangle_{H_k^I} (\beta, \mu_\Lambda^I(\beta,\rho)) = \frac{1}{V} \sum_{n=0}^{+\infty} n\, \nu_{\Lambda,k}(\beta,\rho;n), \tag{A.16}$$

with probability measures: $\{\nu_{\Lambda,k}(\beta,\rho;n)\}_{\Lambda \subset \mathbb{R}^d, k \in \Lambda^*}$:

$$\nu_{\Lambda,k}(\beta,\rho;n) \equiv \frac{e^{-\beta[(\varepsilon_k - \mu_\Lambda^I(\beta,\rho) - \frac{g_k}{2V})\, n + \frac{g_k}{2V} n^2]}}{\sum_{n=0}^{+\infty} e^{-\beta[(\varepsilon_k - \mu_\Lambda^I(\beta,\rho) - \frac{g_k}{2V})\, n + \frac{g_k}{2V} n^2]}}. \tag{A.17}$$

From (A.17) it is clear that we have to distinguish two domains:

$$D_-^{(\Lambda)} \equiv \left\{ k \in \Lambda^* \backslash \{0\} : \varepsilon_k - \mu_\Lambda^I(\beta, \rho) - \frac{g_k}{2V} < 0 \right\},$$
$$D_+^{(\Lambda)} \equiv \left\{ k \in \Lambda^* \backslash \{0\} : \varepsilon_k - \mu_\Lambda^I(\beta, \rho) - \frac{g_k}{2V} \geqslant 0 \right\}, \tag{A.18}$$

cf. (4.3) and (A.2). Since $D_-^{(\Lambda)} \subset \tilde{D}_-^{(\Lambda)}$, our next statement makes the localisation of the *non-extensive* BE condensation more precise, cf. (A.13).

Lemma A.6. For $\rho > \rho_c^I(\beta)$ one has:

$$\tilde{\rho}_0^I(\beta, \rho) = \lim_\Lambda \frac{1}{V} \sum_{k \in D_-^{(\Lambda)}} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) = \rho - \rho_c^I(\beta) > 0. \tag{A.19}$$

Proof. By (A.13) it is sufficient to prove that

$$\lim_\Lambda \frac{1}{V} \sum_{k \in \tilde{D}_-^{(\Lambda)} \backslash D_-^{(\Lambda)}} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) = 0. \tag{A.20}$$

Since $0 < g_- \leqslant g_k \leqslant g_+$, by (A.16) and (A.17) we get that

$$\langle N_k \rangle_{H_\Lambda^I} (\beta, \mu_\Lambda^I(\beta, \rho)) \leqslant \frac{\sum_{n=0}^{+\infty} n e^{-\beta g_k n^2/2V}}{\sum_{n=0}^{+\infty} e^{-\beta g_k n^2/2V}} \frac{\sum_{n=0}^{+\infty} e^{-\beta g_k n^2/2V}}{\sum_{n=0}^{+\infty} e^{-\beta(g_k n/2V + g_k n^2/2V)}} \leqslant O(V^{1/2}) \tag{A.21}$$

for $k \in \tilde{D}_-^{(\Lambda)} \backslash D_-^{(\Lambda)}$ and large V. On the other hand, the set $\tilde{D}_-^{(\Lambda)} \backslash D_-^{(\Lambda)}$ has the same number of elements as the set

$$\left\{ s \equiv \{s_\alpha\}_{\alpha=1}^d \in \mathbb{Z}^d \backslash \{0\} : \frac{g_-}{2V} + \mu_\Lambda^I(\beta, \rho) \leqslant \frac{\hbar^2}{2m} \frac{(2\pi)^2}{V^{2/d}} \sum_{\alpha=1}^d s_\alpha^2 \leqslant \frac{g_+}{V} + \mu_\Lambda^I(\beta, \rho) \right\}. \tag{A.22}$$

Since the volume of the elementary cell of the dual lattice Λ^* is equal to $(2\pi)^d/V$, the number of points (A.22) for large V is *finite*. Together with the estimate (A.21) this gives (A.20). ∎

Theorem A.7. Let $g_k = g > 0$. If $\rho > \rho_c^I(\beta)$, then asymptotics of the chemical potential $\mu_\Lambda^I(\beta, \rho)$ has the form (A.14) with

$$\gamma = 2/(d+2) \quad \text{and} \quad B(\beta, \rho) = \left(\frac{\rho - \rho_c^I(\beta)}{C} \right)^{2/(d+2)}. \tag{A.23}$$

Here $C=(2m/\hbar^2)^{d/2}/[g\,2^{d-2}\pi^{d/2}d(d+2)\,\Gamma(d/2)]>0$, and $\Gamma(z)$ is the Euler gamma function.

Proof. One has to tune the values of $B>0$ and $\gamma>0$ in such a way that to satisfy Eq. (A.19) for $V\to+\infty$. Since we have $\gamma\leqslant 2/d$ (Remark A.5), by using (A.16) and (A.17) we get:

$$\begin{aligned}
&\lim_{\Lambda}\frac{1}{V}\sum_{k\in D_-^{(\Lambda)}}\langle N_k\rangle_{H^I_\Lambda}(\beta,\mu^I_\Lambda(\beta,\rho))\\
&\quad=\lim_{\Lambda}\frac{1}{V}\sum_{\{s\in\mathbb{Z}^d\setminus\{0\}:\frac{\hbar^2}{2m}\frac{(2\pi)^2}{V^{2/d}}\sum_{\alpha=1}^d s_\alpha^2\leqslant g/2V+B/V^\gamma\}}\ \sum_{n=0}^{+\infty} n\nu_{\Lambda,k=2\pi s/V^{1/d}}(\beta,\rho;n)\\
&\quad=\lim_{\Lambda}\frac{1}{V}\sum_{s\in\mathscr{S}_B}\sum_{n=0}^{+\infty} n\nu_{\Lambda,k=2\pi s/V^{1/d-\gamma/2}}\left(V^{1-2\gamma}\beta,\rho;\frac{n}{V^{1-\gamma}}\right)\\
&\quad=\lim_{\Lambda}\left\{\frac{V^{1-\gamma}}{V^{d\gamma/2}}\right\}\frac{1}{V^{1-d\gamma/2}}\sum_{s\in\mathscr{S}_B}\sum_{n=0}^{+\infty}\frac{n}{V^{1-\gamma}}\nu_{\Lambda,k=2\pi s/V^{1/d-\gamma/2}}\left(V^{1-2\gamma}\beta,\rho;\frac{n}{V^{1-\gamma}}\right),
\end{aligned}\tag{A.24}$$

where $\mathscr{S}_B\equiv\{s=\{s_\alpha\}_{\alpha=1}^d\in\mathbb{Z}^d\setminus\{0\}:\frac{\hbar^2}{2m}(2\pi)^2\sum_{\alpha=1}d(s_\alpha/V^{1/d-\gamma/2})^2\leqslant B\}$, see (4.14).

The sum in (A.24) over $\mathscr{S}_B$ is nothing but the Darboux–Riemann sum converging to the integral, when $V\to+\infty$. Therefore, to get a nontrivial limit in (A.24) we must choose the value of γ from the condition: $1-\gamma=d\gamma/2$; i.e., $\gamma=2/(d+2)$ in (A.14). For this value of γ and $d>2$ one has $1-2\gamma>0$. Then the family of the *scaled* probability measures:

$$\{\nu_{\Lambda,k=2\pi s/V^{1/d-\gamma/2}}(V^{1-2\gamma}\beta,\rho;n/V^{1-\gamma})\}_{\Lambda\subset\mathbb{R}^d,\,s\in\mathscr{S}_{\mathscr{B}}},$$

cf. (A.17), verifies the Laplace *large deviation principle.*[28,29] Hence, by a *diagonal* limit involving in (A.24) the sequence of Darboux–Riemann sums and probability measures, and by (A.19), we deduce the equation:

$$\begin{aligned}
\rho-\rho^I_c(\beta)&=\frac{1}{(2\pi)^d}\int_{\{k:\,\varepsilon_k\leqslant B\}}d^dk\,\frac{B-\varepsilon_k}{g_k}\\
&=\{(2m/\hbar^2)^{d/2}/[g\,2^{d-2}\pi^{d/2}d(d+2)\,\Gamma(d/2)]\}\,B^{(d+2)/2},
\end{aligned}\tag{A.25}$$

which defines the value of $B=B(\beta,\rho)$. This finishes the proof of (A.23). ∎

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